PERGAMON

International Journal of Solids and Structures 38 (2001) 8899–8906

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

How to design a foundation

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Received 27 March 2000; in revised form 6 February 2001

Abstract

A foundation is here modelled as long plate of variable thickness placed underneath the base of a long thin wall and resting on the surface of the underlying medium. The foundation distributes the vertical load due to the wall over the relatively soft substrate. The design of the foundation traditionally relies upon empirical or semi-empirical criteria. An analysis of these criteria using optimisation theory reveals that they *probably* depend upon the *principles of minimum stress acting* on the foundation's lower edge and of *maximum permitted height* of the bulb of foundation.

Keywords: Foundations; Shape optimisation

1. Introduction

Assume that a long vertical wall of thickness $2d$ is placed on a deformable half-space (Fig. 1). The wall transmits a vertical load of magnitude p per unit length which is balanced by the substrate. When the thickness becomes small, the normal stress under the base of the wall may become sufficiently large to cause penetration into the half-space. This produces the risk of cracking or even collapse. Settlement is avoided by the common device of inserting a plate underneath the foot of the wall. The structure now has the shape of an inverted T, or, more accurately, of an inverted T for which the junction between the horizontal and vertical parts consists of a symmetric web varying from maximum height H_0 at the centre to minimum height H_1 at the ends (Fig. 1). The respective heights H_0 and H_1 together with the width $2d$ in the past have been chosen according to empirical criteria handed down through successive generations of builders. It is only from the second half of the 18th century that the mathematical procedures have been gradually introduced into the design of foundations. The earliest such methods generally were somewhat simple; see, *e.g.*, the treatise on foundations by Brennecke and Lohmeyer (1948), or the monograph by Haedicke (1967), where stability of the structure depends solely on the approximate evaluation of the maximum tensile stress. Rather than seeking to improve upon the accurate determination of this stress measure, modern treatments have concentrated upon other questions. In particular, a dominant concern has been to decide whether the traditional trapezoidal half cross-section of the foundation can be replaced by another

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PII: S0020-7683(01)00063-4

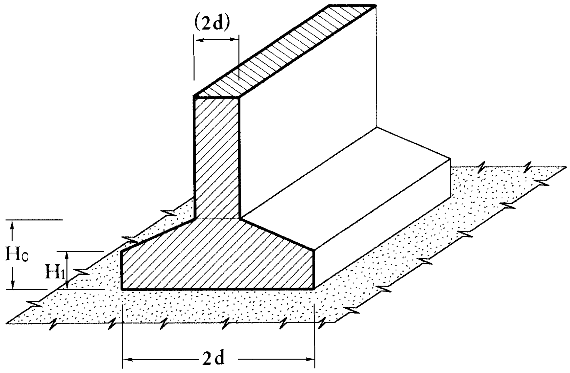

Fig. 1. The typical shape of a foundation having enlarged the thickness of the wall.

symmetric shape of the same area that will either offer greater overall resistance or reduce penetration into the underlying soil.

We propose to discuss this issue for the strip-like symmetric foundation sketched in Fig. 1. Two methods are considered. The first minimises the compliance, defined as the total work performed by the external forces. The second appeals to the criterion of minimum stress intensity which is a measure of maximum stress (see Banichuk (1990, 1.3)). Remarkedly, the dimensions and shape of the traditional, empirically designed, foundation agree reasonably well with those obtained by means of the second optimisation criterion.

2. Optimal solutions with prescribed volume

We require some simplifying assumptions in the formulation of the problem to optimise the shape of the strip-like foundation shown in Fig. 1. We first assume that the thickness $2d$ of the wall is so small with respect to the thickness of the base plate that it can be taken to be zero. The wall then intersects the base in a line and the cross-section in a point. Secondly, we assume that the elastic reaction extorted by the subsoil on the lower face of the base is constant along the length. Although apparently restrictive, this assumption is universally adopted in the design of foundations, since more realistic distributions of pressures obtained by experimental tests on different kinds of soils do not differ too much from the constant distribution (see Haedicke (1967)).

Next, because the lengths of the foundation and the (straight) wall are theoretically infinite, the analysis may be limited to a generic slice of unit thickness. Non-dimensionalisation of variables reduces the problem to the study of a symmetric beam of length 2 subject to unit load uniformly distributed over the base and counterbalanced by a point force of magnitude 2 directed vertically downwards and applied at the midpoint of the opposite edge.

Furthermore, with the system of Cartesian x–y axes indicated in Fig. 2, the lower plane edge of the beam occupies the interval $-1 \leqslant x \leqslant 1$, while the height $h(x)$, due symmetry with respect to the y-axis, is an even function of x. Accordingly, we need consider only the half-beam $0 \leqslant x \leqslant 1$ regarded as built-in at $x = 0$ and subject to a unit load uniformly distributed along the x-axis. The stress resultants in each cross-section may easily be calculated *from elementary theory*. In particular, the bending moment is given by

$$M(x) = \frac{1}{2}(1 - x)^2, \tag{2.1}$$

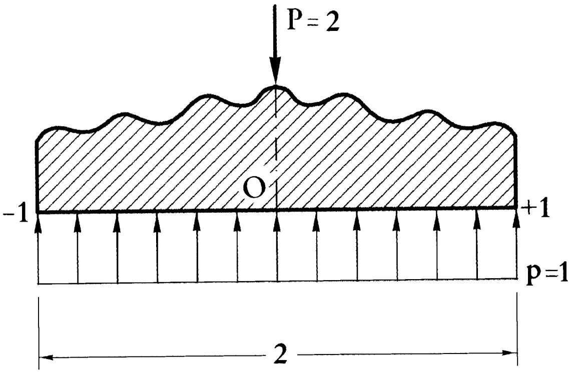

Fig. 2. The foundation considered like a beam (in rescaled units).

and the corresponding curvature by M/EI, where E is Young's modulus and $I(= (1/12)h^3)$ is the moment of inertia of the cross-sectional area. In what follows, we take $E = 1$ for simplicity.

The shape of the beam obviously depends upon the form of the function $h(x)$ which, however, is subject to the condition that the total amount of material used is prescribed. This means that $h(x)$ must satisfy the constraint

$$J_1(h) = \int_0^1 h(x)\,dx = S, \tag{2.2}$$

where S is a specified constant. Without loss, we set $S = 1$ for simplicity.

Once we have chosen $h(x)$ satisfying Eq. (2.2), we can determine the stress and strain state in each cross-section, and, consequently, the total work performed by the external forces, which is known as the "compliance" of the structure, and represents a measure of rigidity of the entire structure. If $\mathscr{L}$ denotes this work, Clapeyron's theorem states that it is given by the relation $\mathscr{L} = 2\mathscr{W}$, where $\mathscr{W}$ is the strain energy, whose expression (having put $E = 1$) is

$$\mathscr{W} = \frac{1}{2}\int_0^1 \frac{M^2(x)}{I(x)}\,dx = \frac{1}{2}J(h(x)), \tag{2.3}$$

with $I = (1/12)h^3(x)$.

The optimisation problem we which to solve consists in finding the function $h(x)$ satisfying the isoperimetric condition (2.2) and minimising the integral (2.3). As is well known, the function $h(x)$ may be found from the extreme values of the functional

$$J(h) + \lambda^2 J_1(h) = \int_0^1 \frac{M^2}{\frac{1}{12}h^3}\,dx + \lambda^2\int_0^1 h\,dx, \tag{2.4}$$

where λ is a Lagrange multiplier. The necessary extremum condition for (2.4) assumes the form (cf. *e.g.* Banichuk (1990, 7.3))

$$h^4(x) = \frac{36M^2}{\lambda^2} = \frac{9(1-x)^4}{\lambda^2}, \tag{2.5}$$

whence we obtain $h(x)$:

$$h(x) = \frac{\sqrt{3}(1-x)}{\sqrt{\lambda}}. \tag{2.6}$$

The Lagrange multiplier λ is evaluated by using the constraint (2.2). The result is

$$\lambda = \frac{3}{4},$$

so that $h(x)$ becomes

$$h(x) = 2(1 - x). \tag{2.7}$$

It follows that under the stated conditions and loads, the optimum shape of the foundation with minimum compliance must be triangular. In addition, we note that the normal stress along the lower edge $0 \leqslant x \leqslant 1$, $y = 0$, calculated according to elementary beam theory, is constant confirming that optimally designed foundations tend to distribute stresses as uniformly as possible. Unfortunately, the traditional shape is not triangular, but consists of a double trapezoid as shown in Fig. 1. We conclude that the practice of earlier architects and engineers has not conformed to the criterion of least compliance.

Let us explore another possible optimisation principle and assume that the shape of the foundation minimises the maximum tensile stress acting on the lower edge. In this case, we must solve the problem

$$\underset{h}{\mathrm{Min}} J = \min_{h} \max_{\substack{0 \leqslant x \leqslant 1 \\ y=0}} |\sigma_x(x, y)|, \tag{2.8}$$

subject to the constraint (2.2). Functional equation (2.8), as distinct from Eq. (2.3), is of local type and to be amenable to standard techniques of the variational calculus requires conversion to an equivalent integral formulation. In accordance with the well-known property of Sobolev spaces (cf. Adams (1975)), this is achieved by replacing J in Eq. (2.7) by its approximation (see Banichuk (1983))

$$J_p = \left(\int_0^1 |\sigma_x(x, 0)|^p \, \mathrm{d}x \right)^{1/p}, \tag{2.9}$$

where $p (p \geqslant 1)$ is a sufficiently large real number. The functional in the extremum problem becomes

$$J_p(h) + \lambda^2 J_1(h) = \left(\int_0^1 |\sigma_x(x, 0)|^p \, \mathrm{d}x \right)^{1/p} + \lambda^2 \int_0^1 h(x) \, \mathrm{d}x, \tag{2.10}$$

where $\sigma_x(x, 0)$ is given by

$$\sigma_x(x, 0) = 6M(x)/h^2(x). \tag{2.11}$$

The first variation of Eq. (2.10) with respect to $h(x)$ yields the equation

$$\frac{2}{\left(\int_0^1 \left(\frac{6M(x)}{h^2(x)} \right)^p \mathrm{d}x \right)^{1-(1/p)}} \frac{(6M(x))^p}{h^{2p+1}(x)} = \lambda^2, \tag{2.12}$$

where $M(x)$ has the expression (2.1). A solution of Eq. (2.12) can be easily obtained by putting

$$h(x) = A(M(x))^{p/(2p+1)}, \tag{2.13}$$

where A is a constant. Substitution of Eq. (2.13) in Eq. (2.2) and subsequent integration gives the result

$$A = \frac{4p + 1}{2p + 1} 2^{2p/(2p+1)}. \tag{2.14}$$

Hence the expression for $h(x)$ becomes

$$h(x) = \frac{4p + 1}{2p + 1} (1 - x)^{2p/(2p+1)}, \tag{2.15}$$

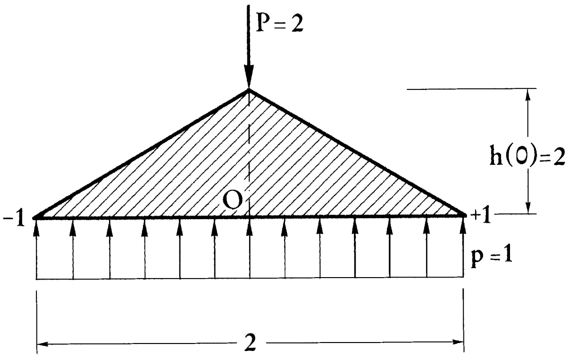

Fig. 3. The foundation of minimum compliance under prescribed area (in rescaled units).

according to a classical property of Sobolev spaces (cf. Adams (1975)). In Eq. (2.15) the number p is still to be chosen. We know that it must be sufficiently large to ensure that the norm J_p is a good approximation to J_∞ the maximum of $|\sigma_x(x,0)|$ for $0 \leqslant x \leqslant 1$. The best value for p obviously is infinite, with the consequence that $h(x)$ in Eq. (2.15) assumes the form (2.7). We arrive at the surprising conclusion that the best foundation is triangular (see Fig. 3) irrespective of whether the compliance or maximum base stress is minimised.

3. Solutions under other constraints

The fact that foundations are not commonly triangular suggests that traditional methods of design have ignored the constraint (2.2) equivalent to specifying the total amount of material employed in the construction. Instead, let us conjecture that perhaps the maximum permissible height of the base has been specified, so that Eq. (2.2) becomes replaced by

$$0 \leqslant h(x) \leqslant h_{\max}, \quad 0 \leqslant x \leqslant 1, \tag{3.1}$$

where $h_{\max}$ denotes the maximum admissible value that is assigned to design variable $h(x)$. However Eq. (3.1) is a local constraint and hence we replace it by the more manageable condition

$$G_p = \left(\int_0^1 |h(x)|^p \, \mathrm{d}x \right)^{1/p} = 1, \tag{3.2}$$

where p is a large real number.

Consider now the problem of minimising the compliance $J(h)$, defined by Eq. (2.3), with the constraint (3.2), or, equivalently, the problem of finding the extrema of the augmented functional

$$J(h) + \lambda^2 G_p(h) = \int_0^1 \frac{M^2}{\frac{1}{12}h^3} \, \mathrm{d}x + \lambda^2 \left(\int_0^1 |h(x)|^p \, \mathrm{d}x \right)^{1/p}, \tag{3.3}$$

λ being a Lagrange multiplier.

By taking the first variation of Eq. (3.3), we obtain the Euler equation

$$36 \frac{M^2(x)}{h^4(x)} = \lambda^2 \left(\int_0^1 |h(x)|^p \, \mathrm{d}x \right)^{(1/p)-1} h^{p-1}(x), \tag{3.4}$$

where $M(x)$ is again given by Eq. (2.1). In order to solve Eq. (3.4) we assume $h(x)$ to have the form

$$h(x) = A(M(x))^{2/(p+3)}, \tag{3.5}$$

where A is a constant determined by Eq. (3.2). Omitting the details of the computation we obtain the value

$$A = 2^{2/(p+3)}\left(\frac{5p+3}{p+3}\right)^{1/p}, \tag{3.6}$$

so that the expression of $h(x)$ is

$$h(x) = \left(\frac{5p+3}{p+3}\right)^{1/p}(1-x)^{4/(p+3)}. \tag{3.7}$$

In this case again, a good approximation of the solution is obtained for large values of p. In the limit, as p tends to infinity, we have the value

$$h(x) \equiv 1, \tag{3.8}$$

so that the height of the optimal foundation is now constant (Fig. 4).

As an alternative to the criterion of minimal compliance we may now apply the criterion of minimum stress along the edge $0 \leqslant x \leqslant 1$, combined with the constraint (3.2). This time, recalling Eq. (2.8), we must find the critical points of the functional

$$J_p(h) + \lambda^2 G_p(h) = \left(\int_0^1 |\sigma_x(x,0)|^p \,\mathrm{d}x\right)^{1/p} + \lambda^2\left(\int_0^1 |h(x)|^p \,\mathrm{d}x\right)^{1/p}, \tag{3.9}$$

where $\sigma_x(x,0)$ is the function of $h(x)$ given by Eq. (2.11) and λ a constant multiplier. The extrema of Eq. (3.9) are solutions of the integral equation

$$\left(\int_0^1 |\sigma_x|^p \,\mathrm{d}x\right)^{(1/p)-1}\frac{(6M)^p}{h^{2p+1}} + \lambda^2\left(\int_0^1 |h|^p \,\mathrm{d}x\right)^{(1/p)-1} h^{p-1} = 0. \tag{3.10}$$

The solution of Eq. (3.10) is of the form

$$h(x) = A(M(x))^{1/3}, \tag{3.11}$$

and the value of A, satisfying Eq. (3.2), is

$$A = 2^{1/3}\left(\frac{3}{2p+3}\right)^{-1/p}, \tag{3.12}$$

so that the expression for $h(x)$ becomes

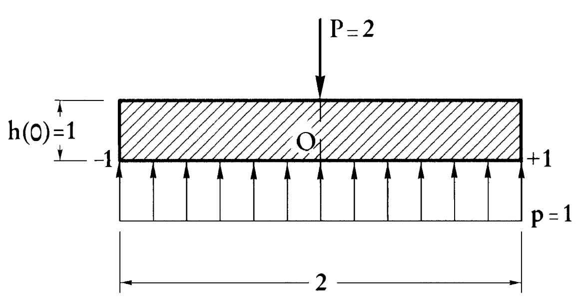

Fig. 4. The foundation of minimum compliance under prescribed height (in rescaled units).

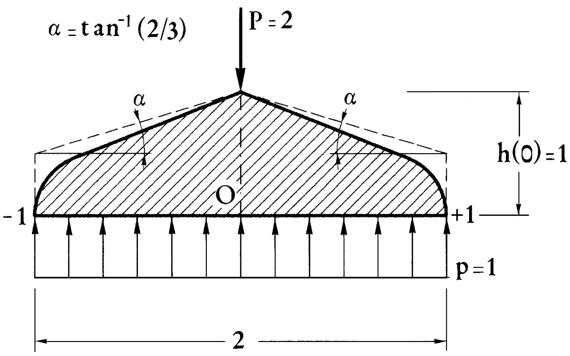

Fig. 5. The foundation of minimum stress under prescribed height (in rescaled units).

$$h(x) = \frac{(1-x)^{2/3}}{\left(\frac{3}{2p+3}\right)^{1/p}}. \tag{3.13}$$

The limit of Eq. (3.12) as p tends to infinity is one. We conclude that $h(x)$, in the limiting case, is given by the equation

$$h(x) = (1-x)^{2/3}, \quad 0 \leqslant x \leqslant 1, \tag{3.14}$$

that is to say $h(x)$ has the shape of the parabola sketched in Fig. 5.

This result helps to explain why foundations are traditionally trapezoidal in shape. Indeed, let us suppose that the parabolic profile determined by Eq. (3.14) is difficult to realise in practice, or is prohibitively expensive. A way of overcoming these drawbacks is to substitute a circumscribed polygon formed by the tangents to the parabola Eq. (3.14), continued by reflection in the y-axis, at the points $(1,0)$, $(-1,0)$ and $(0,1)$. The first two tangents are parallel to the y-axis and vertical. Those at the point $(0,1)$ have slopes $\pm 2/3$. The vertices of the polygon are therefore at $(1,0)$, $(-1,0)$ and at $(1,1/3)$, $(-1,1/3)$. On denoting by α the angle made by the oblique sides with the x-axis, we find that $\alpha = \tan^{-1}(2/3) = 33°6'$, which is comparable to the value of about 28° obtained from the empirical codes (see Leonhardt (1974, Section 16)).

4. Influence of non-constant distributions of pressures

The results obtained so far stem on the hypothesis that the pressure exerted by the subsoil on the base is constant. This assumption, though widely accepted by engineers, is not obvious at all and need a justification. The reason is that, for soils like rock and clay, the distribution of the pressure under the base $-1 \leqslant x \leqslant +1$ has an undulated profile defined by an equation of the type (see Brennecke–Lohmeyer (1948, p. 12))

$$p(x) = A\left(1 - \frac{1}{9}(1-4x^2)^2\right), \tag{4.1}$$

where A is a constant equal, in our case, to $135/112$ in order to satisfy the condition

$$\int_{-1}^{1} p(x)\,\mathrm{d}x = 2. \tag{4.2}$$

The law of distribution reactions Eq. (4.1) represents a reasonable compromise between the case of soft foundation in which the maximal reaction occurs at the middle point $x = 0$ of the base, and that of rigid foundation, in which the reaction is theoretically infinite at the end points $x = \pm 1$ of the base. On the other hand, more than one century of experiments have shown that highest reaction occurs not far from the points $x = \pm\frac{1}{2}$, and that the ends point $x = \pm 1$ are practically unloaded. It is immediate to see that $p(x)$ vanishes at $x = \pm 1$, has a relative minimum at $x = 0$, and two relative maxima at $x = \pm\frac{1}{2}$. The ratio between the values of $p(x)$ at $x = 0$ and $x = \pm\frac{1}{2}$ is $\frac{8}{9}$. This means that $p(x)$ does not differ too much from a constant value, at least on a large central portion of the base.

5. Conclusion

For the strip-like foundation sketched in Fig. 1, the examples we have considered indicate that the shape is strongly influenced by the particular optimisation criteria adopted. The cross-section of the foundation may be triangular, as calculated in Section 2, or rectangular, or parabolic, as found in Section 3. On the other hand, traditional methods appear to have favoured the parabolic profile, or more precisely the polygonal profile approximating to it. Old and modern designs of foundations seem to neglect both the overall compliance of the base and a restriction on the amount of material used in construction. Instead, they rely upon reducing both the maximum stress in foundations (which reputedly were weak) and imposing a maximum height. Risks from a weak substrate are thereby minimised, while at the same time there are accompanying savings on the cost of construction materials. But did the early builders knowingly adopt this recipe?

Acknowledgement

I wish to thank Professor Robin J. Knops for his inestimable suggestions.

References

Adams, R.A., 1975. Sobolev Spaces. Academic Press, New York.
Banichuk, N.V., 1983. Problems and Methods of Optimal Structural Design. Plenum Press, New York.
Banichuk, N.V., 1990. Introduction to Optimization of Structures. Springer, Berlin.
Brennecke, L., Lohmeyer, E., 1948. Der Gründbau. W. Ernst, Berlin.
Haedicke, K., 1967. Gründungen Bd. I. VEB Verlag, Berlin.
Leonhardt, F., 1974. Vorlesungen über Massivbau. Springer, Berlin.